

Maxima & Minima

Grade 12 revision notes

01 / Concepts & tests

1 What is an extremum?

A **local maximum** at $x = c$ means $f(c) \geq f(x)$ for all domain points sufficiently close to c . For a **local minimum**, reverse the inequality.

An **absolute maximum/minimum** is the greatest/least function value over the *entire given domain*. The location is c ; the extreme value is $f(c)$.

Slope and movement

On an interval, $f'(x) > 0 \Rightarrow f$ is increasing;
 $f'(x) < 0 \Rightarrow f$ is decreasing.

Candidates: interior points where $f'(c) = 0$ (stationary points), or where f' does not exist but f is defined. For absolute extrema, also check included endpoints.

2 Why must the slope be zero?

Suppose c is an **interior local maximum** and f is differentiable at c . For sufficiently small h ,

$$f(c+h) - f(c) \leq 0.$$

For $h > 0$, the difference quotient is ≤ 0 ; for $h < 0$, it is ≥ 0 . The two one-sided limits must agree, so

$$f'(c) = 0.$$

The inequalities reverse at a minimum, giving the same result. This is a **necessary condition**, not a sufficient one: $f(x) = x^3$ has $f'(0) = 0$ but no extremum at 0.

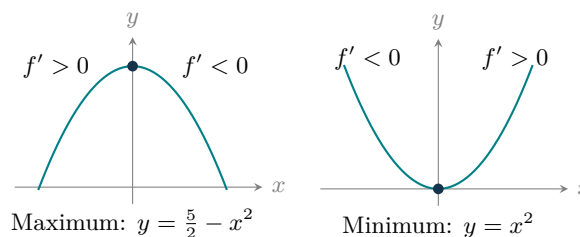
3 First derivative test

Assume f is continuous at c . Inspect the sign of f' just before and after c :

Left	Right	Conclusion at c
+	-	Local maximum
-	+	Local minimum
+	+	Neither
-	-	Neither

Why? Increasing then decreasing makes a peak; decreasing then increasing makes a valley.

4 Read the graphs



5 Second derivative test

For f twice continuously differentiable near c , with $f'(c) = 0$:

$f''(c) < 0 \Rightarrow$ strict local maximum.
 $f''(c) > 0 \Rightarrow$ strict local minimum.
 $f''(c) = 0 \Rightarrow$ **test inconclusive**; use f' signs.

Derivation from the slope: since $f'(c) = 0$,

$$f''(c) = \lim_{h \rightarrow 0} \frac{f'(c+h)}{h}.$$

If this limit is positive, then for small h , $f'(c+h)$ has the sign of h : negative on the left and positive on the right. Thus c is a minimum. A negative limit gives a maximum.

When the test fails: at 0, each of x^4 , $-x^4$, and x^3 has $f''(0) = 0$, yet they give a minimum, a maximum, and neither, respectively.

6 Absolute extrema on $[a, b]$

If f is continuous on a closed, bounded interval $[a, b]$, it attains both absolute extrema.

Method: find all interior critical points; evaluate f there and at a, b ; compare the values. Largest = absolute maximum, smallest = absolute minimum.

Domain matters: $f(x) = x$ on $(0, 1)$ attains neither extreme. Also, $|x|$ has a minimum at 0 even though $f'(0)$ does not exist.

Apply the tests

Worked examples & exam method

02 / Practice

1 A cubic: local extrema

Find the local extrema of $f(x) = x^3 - 3x + 2$.

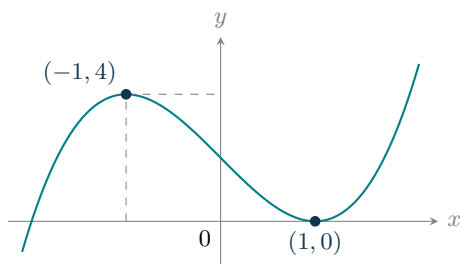
$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1).$$

Stationary points: $x = -1, 1$.

x	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$f'(x)$	+	-	+
f	increasing	decreasing	increasing

Thus $f(-1) = 4$ is a **local maximum**; $f(1) = 0$ is a **local minimum**.

Check: $f''(x) = 6x$; $f''(-1) = -6 < 0$ and $f''(1) = 6 > 0$.



There are **no absolute extrema** on $\mathbb{R}$: the cubic is unbounded above and below.

2 Same curve, restricted domain

Find absolute extrema of $f(x) = x^3 - 3x + 2$ on $[-2, 3]$. Compare endpoints and both critical points:

x	-2	-1	1	3
$f(x)$	0	4	0	20

Absolute maximum: 20 at $x = 3$.

Absolute minimum: 0 at $x = -2$ and $x = 1$. The local maximum 4 is not the absolute maximum on this interval.

3 A stationary point that is neither

For $f(x) = x^3$, $f'(x) = 3x^2$ is positive on both sides of 0. There is no sign change, so 0 is **not an extremum**. The horizontal tangent alone is insufficient.

4 Optimisation: largest rectangle

Problem. A rectangle has perimeter 40 m. Find its greatest possible area.

Let its sides be x and y metres. The constraint gives

$$2(x + y) = 40 \Rightarrow y = 20 - x.$$

For a non-degenerate rectangle, $0 < x < 20$. Express the objective in one variable:

$$A(x) = xy = x(20 - x) = 20x - x^2.$$

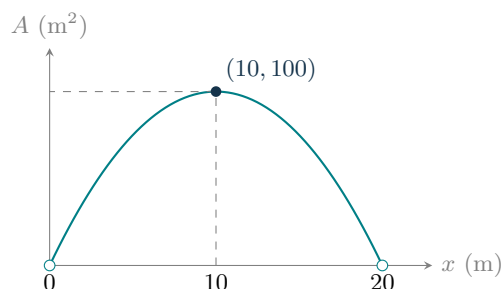
$$A'(x) = 20 - 2x = 0 \Rightarrow x = 10.$$

Since $A''(x) = -2 < 0$, this is a local maximum. To establish it is the greatest possible area,

$$A(x) = 100 - (x - 10)^2 \leq 100.$$

Equality holds at $x = 10$, so $y = 10$.

Answer: a 10 m $\times$ 10 m square;
maximum area = 100 m².



General formula (perimeter P):

$$A = x \left(\frac{P}{2} - x \right), \quad A' = \frac{P}{2} - 2x.$$

$$A' = 0 \Rightarrow x = y = \frac{P}{4}, \quad A_{\max} = \frac{P^2}{16}.$$

5 Exam checklist

- Model:** define variables, units, and domain.
- Reduce:** use the constraint to get one variable.
- Differentiate:** find all valid critical points.
- Justify:** use a sign change or second derivative; compare endpoints for absolute extrema.
- Conclude:** report both location and value, with units and a sentence answering the question.

Quick check. For $g(x) = x^2 - 4x + 5$ on $[0, 3]$, find absolute extrema.

Answer: minimum 1 at $x = 2$; maximum 5 at $x = 0$.
Compare $g(0) = 5$, $g(2) = 1$, $g(3) = 2$.